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Integrals over the circular ensembles relating to classical domains

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Abstract
Using the theory of classical domains and Jack polynomials, we generalize several integrals over the unitary group, due to Fyodorov and Khoruzhenko, to integrals over the circular orthogonal and circular symplectic ensembles. While in the unitary case a fundamental role is played by complex matrices, it is anti-symmetric and symmetric matrices which play the same role in the latter two cases. In our workings, a generalization of the evaluation of the integral of a Jack polynomial times the Selberg weight sought by Fyodorov and Khoruzhenko is identified.

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1. Introduction

Integrals over the unitary ensemble play a prominent role in random matrix theory, the study of disordered systems and in field theory, see [26] and references therein. Under special circumstances such integrals satisfy a type of duality identity, in which the role of the size of the matrix and a parameter in the integrand is reversed. Following up on the recent works [8, 9] relating to the circular unitary ensemble, it is duality integrals of this form, with one side being an integral over the circular orthogonal ensemble, or the circular symplectic ensemble, which are our topic of study.

Let $U(n)$ stands for the group of unitary matrices of size $n \times n$ equipped with the Haar measure dU by the normalization condition $\int_{U(n)} dU = 1$. For any partition λ of length $\ell(\lambda) \leqslant n$, the Schur function S_λ is defined by

$$S_\lambda(x_1, x_2, \ldots, x_n) := \frac{\det\left(x_i^{n-j+\lambda_j}\right)_{i,j=1}^n}{\det\left(x_i^{n-j}\right)_{i,j=1}^n}, \tag{1.1}$$

$S_\lambda(x_1, x_2, \ldots, x_n) = 0$ if $\ell(\lambda) > n$. If Z is an $n \times n$ matrix then the Schur function of the matrix argument is

$$S_\lambda(Z) := S_\lambda(x_1, x_2, \ldots, x_n), \tag{1.2}$$

where $x_1, x_2, \ldots, x_n$ are the eigenvalues of Z. Let λ and μ be partitions and A and B be $n \times n$ matrices, and from the Schur functions of the matrix argument are the characters of irreducible representations of the general linear group and its unitary subgroup, it follows that, see e.g. [16, p 445],

$$\int_{U(n)} S_\lambda(AU)\overline{S_\mu(BU)}\,\mathrm{d}U = \delta_{\lambda,\mu}\frac{S_\lambda(AB^\dagger)}{S_\lambda(1_n)} \tag{1.3}$$

and

$$\int_{U(n)} S_\lambda(AUBU^\dagger)\,\mathrm{d}U = \frac{S_\lambda(A)S_\lambda(B)}{S_\lambda(1_n)}, \tag{1.4}$$

where $U^\dagger$ stands for the conjugate transpose of U, symbol 1_n denotes the n-tuple $(1, 1, \ldots, 1)$.

Let N, m and n be positive integers. Define

$$\mathrm{d}\rho^F_{N,m\times n} = \frac{1}{C}\frac{1}{\det(I + ZZ^\dagger)^{N+m+n}}\,\mathrm{d}m(Z) \tag{1.5}$$

the probability measure on the space $\mathcal{M}_{m,n}(\mathbb{C})$ of complex $m \times n$ matrices. Assume $m \leqslant n$ and $m + n \leqslant N$. Let

$$\mathrm{d}\rho^B_{N,m\times n} = \frac{1}{C}\det(I - ZZ^\dagger)^{N-m-n}\,\mathrm{d}m(Z) \tag{1.6}$$

be the probability measure on the matrix ball

$$\Omega_I(m, n) := \{Z \in \mathcal{M}_{m,n}(\mathbb{C}) : I - ZZ^\dagger > 0, m \leqslant n\}.$$

Here I is the identity matrix and $\mathrm{d}m(Z)$ is the volume element in $\mathcal{M}_{m,n}(\mathbb{C})$,

$$\mathrm{d}m(Z) := \prod_{i=1}^{m}\prod_{j=1}^{n}\mathrm{d}\,\mathrm{Re}Z_{ij}\,\mathrm{d}\,\mathrm{Im}Z_{ij}.$$

By (1.3), Cauchy identities (2.2) and (2.1), and generalized Selberg formulae (2.7) and (2.4) with $\alpha = 1$, Fyodorov and Khoruzhenko have shown that [8, 9]

$$\int_{U(N)} \det(I + AU)^m\overline{\det(I + BU)^n}\,\mathrm{d}U = \int_{\mathcal{M}_{m,n}(\mathbb{C})} \det(I + ZZ^\dagger \otimes AB^\dagger)\,\mathrm{d}\rho^F_{N,m\times n} \tag{1.7}$$

and

$$\int_{U(N)} \frac{\mathrm{d}U}{\det(I - AU)^m\,\overline{\det(I - BU)^n}} = \int_{\Omega_I(m,n)} \frac{\mathrm{d}\rho^B_{N,m\times n}}{\det(I - ZZ^\dagger \otimes AB^\dagger)}. \tag{1.8}$$

Identity (1.8) holds for $A, B \in \Omega_I(N, N)$.

The aim of the present paper is to extend formulae (1.7) and (1.8) to the circular ensembles cases.

The paper is organized as follows. In the following section, we collect basic material about symmetric Jack polynomials, Cauchy identities, generalized Selberg formulae, classical domains, and we will prove that the generalized Selberg formula is equivalent to the dual generalized Selberg formula. Section 3 is dedicated to generalization formulae (1.3) and (1.4) to the Jack function of the matrix argument case. Finally, we derive similar identities of (1.7) and (1.8) in the circular ensembles.

2. Preliminary material

We recall the definition of partitions and Jack symmetric functions, see [16] for details.

2.1. Partitions

Let λ be a partition, i.e., $\lambda := (\lambda_1, \lambda_2, \ldots)$ is a weakly decreasing ordered sequence of non-negative integers with finitely many non-zero entries. The length and weight of λ, denoted by $\ell(\lambda)$ and $|\lambda|$, are the number and sum of non-zero λ_i, respectively. If $|\lambda| = n$, then we write $\lambda \vdash n$. The partition λ can also be written in the form $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots)$, where $m_i(\lambda)$ is the multiplicity of i in λ, so $|\lambda| = \sum_i i m_i(\lambda)$, $\ell(\lambda) = \sum_i m_i(\lambda)$.

For two partitions λ and ν, we set $\lambda + \nu := (\lambda_1 + \nu_1, \lambda_2 + \nu_2, \ldots)$, $2\lambda := \lambda + \lambda$, $\lambda \cup \lambda := (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)$ and $\lambda + a := (\lambda_1 + a, \lambda_2 + a, \ldots)$, here a is an integer.

The dominance partial order on the set of partitions is defined as follows: for partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$, $\lambda \geqslant \nu$ if and only if $|\lambda| = |\nu|$ and $\lambda_1 + \cdots + \lambda_i \geqslant \nu_1 + \cdots + \nu_i$ for all $i \geqslant 1$.

We identify a partition with its diagram or Ferrers graph, defined by $\{(i, j) \in \mathbb{N}^2 | 1 \leqslant j \leqslant \lambda_i\}$. The conjugate λ^t of λ is the partition obtained by reflecting the diagram of λ in the main diagonal. Let $s = (i, j)$ be a square in the diagram of λ, then $a(s), a'(s), l(s)$ and $l'(s)$ are the arm length, arm colength, leg length and leg colength of s, defined by $a(s) := \lambda_i - j$, $a'(s) := j - 1$, $l(s) := \lambda^t_j - i$ and $l'(s) := i - 1$, respectively.

2.2. Symmetric Jack polynomials

For each $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ we denote by x^γ the monomial

$$x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n}.$$

Let λ be any partition of length $\leqslant n$, the polynomial

$$m_\lambda(x_1, \ldots, x_n) := \sum x^\gamma$$

summed over all distinct permutations γ of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is symmetric, and called the monomial symmetric function.

For each $r \geqslant 1$ the rth power sum is

$$p_r := \sum x_i^r.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we define a power sum symmetric function as

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots.$$

Let α be a positive constant, the bilinear scalar product $\langle \cdot, \cdot \rangle_\alpha$ on the vector space of homogeneous symmetric functions is defined by

$$\langle p_\lambda, p_\nu \rangle_\alpha = \delta_{\lambda\nu} z_\lambda \alpha^{\ell(\lambda)},$$

where $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots)$, $z_\lambda := \prod_{i \geqslant 1} i^{m_i(\lambda)} m_i(\lambda)!$.

The symmetric Jack polynomials $P_\lambda^{(\alpha)}$ are the unique symmetric polynomial such that

$$P_\lambda^{(\alpha)} = m_\lambda + \sum_{\nu < \lambda} c_{\lambda\nu}^{(\alpha)} m_\nu$$

and

$$\left\langle P_\lambda^{(\alpha)}, P_\nu^{(\alpha)} \right\rangle_\alpha = 0,$$

if $\lambda \neq \nu$.

For $\alpha = 1$ and a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, the polynomials $P_\lambda^{(\alpha)}(x_1, x_2, \dots, x_n)$ called the Schur function are denoted by

$$S_\lambda(x_1, x_2, \dots, x_n) := \frac{\det\left(x_i^{n-j+\lambda_j}\right)_{i,j=1}^n}{\det\left(x_i^{n-j}\right)_{i,j=1}^n}.$$

With the appropriate normalization, the symmetric Jack polynomials $P_\lambda^{(\alpha)}$ are the zonal polynomials when $\alpha = 2, \frac{1}{2}$.

2.3. *Cauchy identities and generalized Selberg formulae*

To make the paper self-contained we state the following lemmas.

For $\alpha > 0$, and a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, define

$$d_\lambda(\alpha) := \prod_{s\in\lambda}(\alpha a(s) + \alpha + l(s) + 1),$$

$$d'_\lambda(\alpha) := \prod_{s\in\lambda}(\alpha a(s) + \alpha + l(s)),$$

$$e_\lambda(\alpha, n) := \prod_{s\in\lambda}(\alpha a'(s) + \alpha + n - l'(s)),$$

$$e'_\lambda(\alpha, n) := \prod_{s\in\lambda}(\alpha a'(s) + \alpha + n - 1 - l'(s)),$$

$$h_\lambda(\alpha) := \prod_{s\in\lambda}(\alpha a(s) + l(s) + 1),$$

$$b_\lambda(\alpha, n) := \prod_{s\in\lambda}(\alpha a'(s) + n - l'(s))$$

and

$$[r]_\lambda^{(\alpha)} := \prod_j \frac{\Gamma\left(r - \frac{j-1}{\alpha} + \lambda_j\right)}{\Gamma\left(r - \frac{j-1}{\alpha}\right)}.$$

Then, we have

$$e_\lambda(\alpha, n) = \alpha^{|\lambda|}\left[1 + \frac{n}{\alpha}\right]_\lambda^{(\alpha)}, \qquad e'_\lambda(\alpha, n) = \alpha^{|\lambda|}\left[1 + \frac{n-1}{\alpha}\right]_\lambda^{(\alpha)}$$

and

$$b_\lambda(\alpha, n) = \alpha^{|\lambda|}\left[\frac{n}{\alpha}\right]_\lambda^{(\alpha)}.$$

Lemma 2.1 (Cauchy identities [7]). *Let* $x = (x_1, x_2, \dots, x_m)$, $y = (y_1, y_2, \dots, y_n)$ *and* $\alpha > 0$. *We have Cauchy identity*

$$\prod_{i=1}^m \prod_{j=1}^n \frac{1}{(1 - x_i y_j)^{\frac{1}{\alpha}}} = \sum_\lambda \frac{h_\lambda(\alpha)}{d'_\lambda(\alpha)} P_\lambda^{(\alpha)}(x) P_\lambda^{(\alpha)}(y), \tag{2.1}$$

and dual Cauchy identity

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_iy_j)=\sum_{\lambda}P_{\lambda}^{(\alpha)}(x)P_{\lambda^t}^{(\frac{1}{\alpha})}(y). \tag{2.2}$$

Lemma 2.2 (Generalized binomial summation formula [7, 13, 24])**.** *Let $x_1, x_2, \ldots, x_n$ be real numbers such that $|x_i| < 1$ for $1 \leqslant i \leqslant n$, then*

$$\frac{1}{\prod_{i=1}^{n}(1-x_i)^s}=\sum_{\lambda}\frac{\alpha^{|\lambda|}[s]_{\lambda}^{(\alpha)}}{d_{\lambda}'(\alpha)}P_{\lambda}^{(\alpha)}(x_1,x_2,\ldots,x_n). \tag{2.3}$$

Lemma 2.3 (Generalized Selberg formula [7, 10, 11, 16])**.** *Let λ be any partition of length $\leqslant n$, and $\mathrm{Re}(x) > -1, \mathrm{Re}(y) > -1, \alpha > 0$, we have*

$$\int_{[0,1]^n}P_{\lambda}^{(\alpha)}(t_1,t_2,\ldots,t_n)\prod_{i=1}^{n}t_i^x(1-t_i)^y\prod_{1\leqslant j<k\leqslant n}|t_j-t_k|^{\frac{2}{\alpha}}\prod_{j=1}^{n}\mathrm{d}t_j$$
$$=P_{\lambda}^{(\alpha)}(1_n)\frac{\left[x+1+\frac{n-1}{\alpha}\right]_{\lambda}^{(\alpha)}}{\left[x+y+2+2\frac{n-1}{\alpha}\right]_{\lambda}^{(\alpha)}}S_n(x,y;\alpha), \tag{2.4}$$

where Selberg integral (see [1, 7])

$$S_n(x,y;\alpha):=\int_{[0,1]^n}\prod_{i=1}^{n}t_i^x(1-t_i)^y\prod_{1\leqslant j<k\leqslant n}|t_j-t_k|^{\frac{2}{\alpha}}\prod_{j=1}^{n}\mathrm{d}t_j$$
$$=\prod_{j=0}^{n-1}\frac{\Gamma\left(x+1+\frac{j}{\alpha}\right)\Gamma\left(y+1+\frac{j}{\alpha}\right)\Gamma\left(1+\frac{j+1}{\alpha}\right)}{\Gamma\left(x+y+2+\frac{n+j-1}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)}. \tag{2.5}$$

$P_{\lambda}^{(\alpha)}(1_n)$ denotes the value of a function $P_{\lambda}^{(\alpha)}(t_1,t_2,\ldots,t_n)$ at $(t_1,t_2,\ldots,t_n)=(1,1,\ldots,1)$ and

$$P_{\lambda}^{(\alpha)}(1_n)=\frac{b_{\lambda}(\alpha,n)}{h_{\lambda}(\alpha)}. \tag{2.6}$$

Lemma 2.4 (Dual generalized Selberg formula [9, 23])**.** *Let x, y be complex numbers, α be a positive number, and λ be a partition of length $\leqslant n$, such that the length of its conjugate partition λ^t is less than $\mathrm{Re}(y)+1$, and $\mathrm{Re}(x) > -1, \mathrm{Re}(y) > -1, \alpha > 0$, we have*

$$\int_{[0,+\infty)^n}P_{\lambda}^{(\alpha)}(t_1,t_2,\ldots,t_n)\prod_{i=1}^{n}t_i^x(1+t_i)^{-x-y-2-2\frac{n-1}{\alpha}}\prod_{1\leqslant j<k\leqslant n}|t_j-t_k|^{\frac{2}{\alpha}}\prod_{j=1}^{n}\mathrm{d}t_j$$
$$=P_{\lambda}^{(\alpha)}(1_n)\frac{\left[x+1+\frac{n-1}{\alpha}\right]_{\lambda}^{(\alpha)}}{(-1)^{|\lambda|}[-y]_{\lambda}^{(\alpha)}}S_n(x,y;\alpha), \tag{2.7}$$

where $S_n(x,y;\alpha)$ as in (2.5).

Remark 2.1. This companion to (2.4) was proved in the case $\alpha = 1$ in [9], and the question posed as to its Jack generalization, which is our (2.7).

In the following, we give alternative proof of lemma 2.4.

Proof of lemma 2.4. First we show that (2.7) holds for positive integers $\frac{1}{\alpha}$.

To prove lemma 2.4, we need formula (2.11) which is due to [10].

Let $\frac{1}{\alpha}$ be a positive integer, and setting

$$P_\lambda^{(\alpha)}(t_1, t_2, \ldots, t_n) \prod_{1\leqslant i<j\leqslant n} |t_i - t_j|^{\frac{2}{\alpha}} = \sum_\eta c_{\lambda\eta}(n,\alpha) \prod_{i=1}^n t_i^{\eta_i}, \tag{2.8}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ and integers $\eta_i \geqslant 0$. By the homogeneity of the Jack symmetric functions, we have

$$\sum_{i=1}^n \eta_i = |\lambda| + \frac{n(n-1)}{\alpha}. \tag{2.9}$$

Substituting (2.8) into (2.4) and carrying out each beta integral separately give

$$\sum_\eta c_{\lambda\eta}(n,\alpha) \prod_{i=1}^n \frac{\Gamma(x+1+\eta_i)\Gamma(y+1)}{\Gamma(x+y+2+\eta_i)} = P_\lambda^{(\alpha)}(1_n) \frac{\left[x+1+\frac{(n-1)}{\alpha}\right]_\lambda^{(\alpha)}}{\left[x+y+2+2\frac{(n-1)}{\alpha}\right]_\lambda^{(\alpha)}} S_n(x,y;\alpha). \tag{2.10}$$

Dividing by $(B(x+1,y+1))^n \equiv \left(\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}\right)^n$, we have

$$\sum_\eta c_{\lambda\eta}(n,\alpha) \prod_{i=1}^n \frac{(x+1)_{\eta_i}}{(x+y+2)_{\eta_i}} = \frac{P_\lambda^{(\alpha)}(1_n)}{(B(x+1,y+1))^n} \frac{\left[x+1+\frac{(n-1)}{\alpha}\right]_\lambda^{(\alpha)}}{\left[x+y+2+2\frac{(n-1)}{\alpha}\right]_\lambda^{(\alpha)}} S_n(x,y;\alpha). \tag{2.11}$$

where $(x)_k := \frac{\Gamma(x+k)}{\Gamma(x)}$. Observe that this holds for all x and y since both sides are rational functions for x and y.

Now substituting (2.8) into the lhs of (2.7) and carrying out each integral separately give

$$\text{lhs of (2.7)} = \left(B\left(x+1, y+1+2\frac{(n-1)}{\alpha}\right)\right)^n \sum_\eta c_{\lambda\eta}(n,\alpha)(-1)^{\sum_{j=1}^n \eta_j} \prod_{i=1}^n \frac{(x+1)_{\eta_i}}{\left(-y-2\frac{(n-1)}{\alpha}\right)_{\eta_i}}. \tag{2.12}$$

By (2.9) and (2.11), we have

$$\text{lhs of (2.7)} = \left(\frac{B\left(x+1, y+1+2\frac{(n-1)}{\alpha}\right)}{B\left(x+1, -x-y-1-2\frac{n-1}{\alpha}\right)}\right)^n \times P_\lambda^{(\alpha)}(1_n) \frac{\left[x+1+\frac{n-1}{\alpha}\right]_\lambda^{(\alpha)}}{(-1)^{|\lambda|}[-y]_\lambda^{(\alpha)}} S_n\left(x, -x-y-2-2\frac{n-1}{\alpha}; \alpha\right). \tag{2.13}$$

For $\lambda = 0$, using $[x]_\lambda^{(\alpha)} = P_\lambda^{(\alpha)}(1_n) = 1$, (2.13) imply that

$$\int_{[0,+\infty)^n} \prod_{i=1}^n t_i^a (1+t_i)^{-x-y-2-2\frac{n-1}{\alpha}} \prod_{1\leqslant j<k\leqslant n} |t_j - t_k|^{\frac{2}{\alpha}} \prod_{j=1}^n \mathrm{d}t_j = \left(\frac{B\left(x+1, y+1+2\frac{(n-1)}{\alpha}\right)}{B\left(x+1, -x-y-1-2\frac{n-1}{\alpha}\right)}\right)^n S_n\left(x, -x-y-2-2\frac{n-1}{\alpha}; \alpha\right). \tag{2.14}$$

From Selberg formula we know that

$$\int_{[0,+\infty)^n} \prod_{i=1}^n t_i^a (1+t_i)^{-x-y-2-2\frac{n-1}{\alpha}} \prod_{1\leqslant j<k\leqslant n} |t_j - t_k|^{\frac{2}{\alpha}} \prod_{j=1}^n \mathrm{d}t_j = S_n(x,y;\alpha). \tag{2.15}$$

Comparing (2.14) and (2.15), we have

$$S_n(x,y;\alpha) = \left(\frac{B\left(x+1, y+1+2\frac{(n-1)}{\alpha}\right)}{B\left(x+1, -x-y-1-2\frac{n-1}{\alpha}\right)}\right)^n S_n\left(x, -x-y-2-2\frac{n-1}{\alpha};\alpha\right). \quad (2.16)$$

Substituting (2.16) into (2.13) we obtain (2.7).

Next we prove that (2.7) holds for positive numbers $\frac{1}{\alpha}$.

Set $k := \frac{1}{\alpha}$,

$$g(k) := \int_{[0,+\infty)^n} P_\lambda^{(\frac{1}{k})}(t_1,t_2,\ldots,t_n)\prod_{i=1}^n t_i^x(1+t_i)^{-x-y-2-2\frac{n-1}{\alpha}}\prod_{1\leqslant j<k\leqslant n}|t_j-t_k|^{2k}\prod_{j=1}^n \mathrm{d}t_j, \quad (2.17)$$

$$h(k) := P_\lambda^{(\frac{1}{k})}(1_n)\frac{[x+1+k(n-1)]_\lambda^{(\frac{1}{k})}}{(-1)^{|\lambda|}[-y]_\lambda^{(\frac{1}{k})}} S_n\left(x,y;\frac{1}{k}\right), \quad (2.18)$$

and $f(k) := g(k) - h(k)$.

From [10], Jack symmetric function $P_\lambda^{(\frac{1}{k})}$ can be written as

$$P_\lambda^{(\frac{1}{k})}(t_1,t_2,\ldots,t_n) = \sum_\eta C_{\lambda\eta}(k)\prod_{i=1}^n t_i^{\eta_i}, \quad (2.19)$$

where $\eta = (\eta_1,\eta_2,\ldots,\eta_n)$, $\eta_i \geqslant 0$, $\max_{1\leqslant j\leqslant n}\eta_i \leqslant \lambda_1$, $\sum_{i=1}^n \eta_i = |\lambda|$, $C_{\lambda\eta}(k)$ are rational functions in k, and there exist constant numbers $M_{\lambda\eta}$ such that

$$|C_{\lambda\eta}(k)| \leqslant M_{\lambda\eta}, \qquad \mathrm{Re}(k) \geqslant 1. \quad (2.20)$$

Substituting (2.19) into (2.17), and using the variables change $t_i \to \frac{t_i}{1-t_i}$ we get

$$g(k) = \sum_\eta C_{\lambda\eta}(k)\int_{[0,1]^n}\prod_{j=1}^n t_j^{x+\eta_j}(1-t_j)^{y-\eta_j}\prod_{1\leqslant i<j\leqslant n}|t_i-t_j|^{2k}\prod_{i=1}^n \mathrm{d}t_i, \quad (2.21)$$

and (2.21) implies that $g(k)$ is the analytic function in k when $\mathrm{Re}(k) > 0$. When $\mathrm{Re}(k) \geqslant 1$, by (2.20) and (2.21) we have

$$|g(k)| \leqslant \sum_\eta M_{\lambda\eta}\int_{[0,1]^n}\prod_{j=1}^n t_j^{\mathrm{Re}(x)+\eta_j}(1-t_j)^{\mathrm{Re}(y)-\eta_j}\prod_{i=1}^n \mathrm{d}t_i. \quad (2.22)$$

Let

$$q(k) := P_\lambda^{(\frac{1}{k})}(1_n)\frac{[x+1+k(n-1)]_\lambda^{(\frac{1}{k})}}{(-1)^{|\lambda|}[-y]_\lambda^{(\frac{1}{k})}}. \quad (2.23)$$

From the definition of symbol $[x]_\lambda^{(\alpha)}$ and (2.6), we know that $q(k)$ is a rational function in k, and $\lim_{|k|\to\infty} q(k)$ exists. By (2.5), when $\mathrm{Re}(k) \geqslant 1$ we have

$$S_n\left(x,y;\frac{1}{k}\right) \leqslant \int_{[0,1]^n}\prod_{j=1}^n t_j^{\mathrm{Re}(x)}(1-t_j)^{\mathrm{Re}(y)}\prod_{i=1}^n \mathrm{d}t_i.$$

This indicates that there exists constant N such that

$$|h(k)| = |q(k)|S_n\left(x,y;\frac{1}{k}\right) \leqslant N, \qquad \mathrm{Re}(k) \geqslant 1. \quad (2.24)$$

From (2.22) and (2.24), we obtain that $f(k)$ is bounded for $\mathrm{Re}(k) \geqslant 1$. Since for positive integers k, $f(k) = 0$, by Carlson's theorem [21], we have $f(k) = 0$ for $\mathrm{Re}(k) \geqslant 1$.

Finally, since $f(k)$ is analytic for $\mathrm{Re}(k) > 0$, it follows that $f(k) = 0$ identically. This proves lemma 2.4. □

Remark 2.2. By (2.7), we have (2.11), this means that lemma 2.3 is equivalent to lemma 2.4.

Lemma 2.3 implies the following result.

Lemma 2.5. *Let λ be any partition of length $\leqslant n$, $\mathrm{Re}(x) > -1$, $s > 0$ and $\alpha > 0$. We have*

$$\int_{[0,+\infty)^n} P_\lambda^{(\alpha)}(t_1, t_2, \ldots, t_n) \prod_{i=1}^n t_i^x \exp\{-st_i\} \prod_{1\leqslant j<k\leqslant n} |t_j - t_k|^{\frac{2}{\alpha}} \prod_{j=1}^n \mathrm{d}t_j$$
$$= \frac{1}{s^{|\lambda|}} P_\lambda^{(\alpha)}(1_n) \left[x + 1 + \frac{n-1}{\alpha}\right]_\lambda^{(\alpha)} \int_{[0,+\infty)^n} \prod_{i=1}^n t_i^x \exp\{-st_i\} \prod_{1\leqslant j<k\leqslant n} |t_j - t_k|^{\frac{2}{\alpha}} \prod_{j=1}^n \mathrm{d}t_j. \tag{2.25}$$

2.4. Classical domains

Let Ω be an irreducible bounded symmetric domain (Cartan domain) in $\mathbb{C}^d$ in its Harish–Chandra realization, for convenience, the following setting $\Omega^D := \mathbb{C}^d$. Thus Ω is the open unit ball of a Banach space which admits the structure of a JB^*-triple. We denote by r, a, b, d, g and $N(Z, \overline{W})$ the rank, the characteristic multiplicities, the dimension, the genus and the generic norm of Ω, respectively; thus

$$d = \frac{r(r-1)}{2}a + rb + r, \qquad g = (r-1)a + b + 2. \tag{2.26}$$

For any $s > -1$, the value of the Hua integral $\int_\Omega N(Z, \overline{Z})^s \,\mathrm{d}m(Z)$ is given by

$$\int_\Omega N(Z, \overline{Z})^s \,\mathrm{d}m(Z) = \frac{\chi(0)}{\chi(s)} \int_\Omega \mathrm{d}m(Z), \tag{2.27}$$

where $\mathrm{d}m(Z)$ denotes the Lebesgue measure on $\mathbb{C}^d$, χ is the Hua polynomial

$$\chi(s) := \prod_{j=1}^r \left(s + 1 + (j-1)\frac{a}{2}\right)_{1+b+(r-j)a}. \tag{2.28}$$

Here $(s)_m$ denotes the raising factorial

$$(s)_m := \frac{\Gamma(s+m)}{\Gamma(s)}.$$

Let $\mathcal{G}$ stands for the identity connected component of the group of biholomorphic self-maps of Ω, $\mathcal{K}$ for stabilizer of the origin in $\mathcal{G}$.

Let $e_1, e_2, \ldots, e_r \in \mathbb{C}^d$ be a Jordan frame, for each $Z \in \mathbb{C}^d$ has the polar decomposition

$$Z = k\cdot(t_1e_1 + t_2e_2 + \cdots + t_re_r), \qquad k \in \mathcal{K}, \quad t_1 \geqslant t_2 \geqslant \cdots \geqslant t_r \geqslant 0, \tag{2.29}$$

the numbers $t_1, t_2, \ldots, t_r$ are called the singular values of Z.

Let $\partial_0\Omega$ be the Shilov boundary of Ω, i.e.

$$\partial_0\Omega := \{k\cdot e : k \in \mathcal{K}\},$$

where $e = \sum_{j=1}^r e_j$.

Under the action $f \mapsto f\circ k (k \in \mathcal{K})$ of $\mathcal{K}$, the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^d$ admits the Peter–Weyl decomposition

$$\mathcal{P} = \bigoplus_\lambda \mathcal{P}_\lambda,$$

where the summation is over all partitions λ, i.e. r-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of nonnegative integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$, the spaces $\mathcal{P}_\lambda$ are $\mathcal{K}$-invariant and irreducible. Let $\chi_\lambda(k)$ be characters of irreducible representations of $\mathcal{K}$ on $\mathcal{P}_\lambda$, then

$$\dim \mathcal{P}_\lambda = \chi_\lambda(I),$$

where I denotes the identity element of $\mathcal{K}$. For each $\lambda, \mathcal{P}_\lambda \subset \mathcal{P}_{|\lambda|}$, where $|\lambda|$ denotes the weight of partition λ, i.e. $|\lambda| := \sum_{j=1}^{r} \lambda_j$, and $\mathcal{P}_{|\lambda|}$ is the space of homogeneous polynomials of degree $|\lambda|$.

Let

$$\langle f, g\rangle_{\mathcal{F}} := \int_{\Omega^D} f(Z)\overline{g(Z)}\,\mathrm{d}\rho_{\mathcal{F}}(Z) \tag{2.30}$$

be the Fock–Fischer inner product on the space $\mathcal{P}$ of polynomials, where

$$\mathrm{d}\rho_{\mathcal{F}}(Z) := \frac{1}{\pi^d}\exp\{-|Z|^2\}\,\mathrm{d}m(Z). \tag{2.31}$$

For every partition λ and number $\alpha := \frac{2}{a}$, let $K_\lambda^{(\alpha)}(Z_1, \overline{Z_2})$ be the reproducing kernel of $\mathcal{P}_\lambda$ with respect to (2.30). The weight Bergman kernel (reproducing kernel) of the weight Bergman space $L_a^2(\mathbb{C}^d, \rho_{\mathcal{F}})$ (i.e. the reproducing kernel of the Hilbert space of square integrable analytic functions with respect to the measure $\rho_{\mathcal{F}}$ on $\mathbb{C}^d$) is

$$K(Z_1, \overline{Z_2}) := \sum_\lambda K_\lambda^{(\alpha)}(Z_1, \overline{Z_2}) = \exp\{\langle Z_1, Z_2\rangle\}. \tag{2.32}$$

The kernels $K_\lambda^{(\alpha)}(Z_1, \overline{Z_2})$ are related to the generic norm $N(Z_1, \overline{Z_2})$ by the Faraut–Koranyi formula

$$N(Z_1, \overline{Z_2})^{-s} = \sum_\lambda [s]_\lambda^{(\alpha)} K_\lambda^{(\alpha)}(Z_1, \overline{Z_2}), \tag{2.33}$$

the series converges uniform on compact subsets of $\Omega \times \Omega$, $s \in \mathbb{C}$, where $[s]_\lambda^{(\alpha)}$ denotes the generalized Pochhammer symbol

$$[s]_\lambda^{(\alpha)} := \prod_{j=1}^{r}\left(s - \frac{j-1}{2}a\right)_{\lambda_j}. \tag{2.34}$$

For Z as in (2.29), it is known that

$$K_\lambda^{(\alpha)}(Z, \overline{Z}) = K_\lambda^{(\alpha)}\left(\sum_{j=1}^{r} t_j^2 e_j, \overline{e}\right), \tag{2.35}$$

$$N(Z, \overline{Z}) = \prod_{j=1}^{r}\left(1 - t_j^2\right), \tag{2.36}$$

$$N_+(Z, \overline{Z}) := N(-Z, \overline{Z}) = \prod_{j=1}^{r}\left(1 + t_j^2\right). \tag{2.37}$$

By (2.3), (2.33), (2.35) and (2.36), one has

$$K_\lambda^{(\alpha)}\left(\sum_{j=1}^{r} t_j e_j, \overline{e}\right) = \frac{\alpha^{|\lambda|}}{d'_\lambda(\alpha)} P_\lambda^{(\alpha)}(t_1, t_2, \ldots, t_r), \tag{2.38}$$

where $\alpha = \frac{2}{a}$.

Since this paper only involves Cartan domains (classical domains) of first, second and third types, in the following we list the definitions, the generic norms $N(Z, \overline{W})$, the numerical invariants r, a, b, p and the Shilov boundaries $\partial_0\Omega$ of these domains.

Let

$$\mathcal{M}_{m,n}(\mathbb{C}) := \{m \times n \text{ complex matrices } Z\},$$
$$\mathcal{S}_n(\mathbb{C}) := \{Z \in \mathcal{M}_{n,n}(\mathbb{C}) : Z^t = Z\},$$
$$\mathcal{A}_n(\mathbb{C}) := \{Z \in \mathcal{M}_{n,n}(\mathbb{C}) : Z^t = -Z\},$$

where symbol Z^t stands for the transpose of Z. Let $\Omega_I(m,n)$, $\Omega_{II}(n)$ and $\Omega_{III}(n)$ denote classical domains of first, second and third types in the sense of Hua, respectively, i.e.

$$\Omega_I(m,n) := \{Z \in \mathcal{M}_{m,n}(\mathbb{C}) : I - ZZ^\dagger > 0, m \leqslant n\},$$
$$\Omega_{II}(n) := \{Z \in \mathcal{S}_n(\mathbb{C}) : I - ZZ^\dagger > 0\},$$
$$\Omega_{III}(n) := \{Z \in \mathcal{A}_n(\mathbb{C}) : I - ZZ^\dagger > 0\},$$

where $Z^\dagger$ stands for the conjugate transpose of Z, $I - ZZ^\dagger > 0$ denotes $I - ZZ^\dagger > 0$ is definite positive. The invariants r, a, b, g, d, the generic norm $N(Z, \overline{Z})$, group $\mathcal{K}$ and characters $\chi_\lambda(k)$ of these domains are summarized in the table below.

Domain	r	a	b	d	g	$N(Z,\overline{Z})$	$\mathcal{K}$	$\chi_\lambda(k)$
$\Omega_I(m,n)$	m	2	$n-m$	mn	$m+n$	$\det(I-ZZ^\dagger)$	$U(m)\otimes U(n)$	$S_\lambda(A)S_\lambda(B)$
$\Omega_{II}(n)$	n	1	0	$\frac{n(n+1)}{2}$	$n+1$	$\det(I-ZZ^\dagger)$	$U(n)$	$S_{2\lambda}(A)$
$\Omega_{III}(2n)$	n	4	0	$n(2n-1)$	$2(n-1)$	$\sqrt{\det(I-ZZ^\dagger)}$	$U(2n)$	$S_{\lambda\cup\lambda}(A)$
$\Omega_{III}(2n+1)$	n	4	2	$n(2n+1)$	$2(n-1)$	$\sqrt{\det(I-ZZ^\dagger)}$	$U(2n+1)$	$S_{\lambda\cup\lambda}(A)$

Finally, we list the Shilov boundaries of these domains as follows:

$$\partial_0\Omega_I(m,n) := \{U \in \mathcal{M}_{m,n}(\mathbb{C}) : UU^\dagger = I\},$$
$$\partial_0\Omega_{II}(n) := \{U \in \mathcal{S}_n(\mathbb{C}) : UU^\dagger = I\},$$
$$\partial_0\Omega_{III}(2n) := \{U \in \mathcal{A}_{2n}(\mathbb{C}) : UU^\dagger = I\},$$
$$\partial_0\Omega_{III}(2n+1) := \{U(Z_{2n} \oplus 0)U^t : UU^\dagger = U^\dagger U = I\},$$

where

$$Z_{2n} := I_n \otimes Z_2, \qquad Z_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the proofs of above facts and additional details, we refer e.g. to [5, 13, 15, 22, 25].

Remark 2.3. If $b = 0$, Cartan domain Ω is called tube type domain. Of the third tube type, an irreducible bounded symmetric domain can be realized as

$$\Omega_{III}(2n) := \{Z \in \mathcal{D}_n(\mathbb{C}) : I - ZZ^\dagger > 0\},$$

where

$$\mathcal{D}_n(\mathbb{C}) := \{Z \in \mathcal{M}_{2n,2n}(\mathbb{C}) : Z^D = Z\},$$
$$Z^D := Z_{2n}^{-1} Z^t Z_{2n}.$$

Then the Shilov boundaries of tube type domains $\Omega_I(n,n)$, $\Omega_{II}(n)$ and $\Omega_{III}(2n)$ are the circular ensembles $U(n)$, $COE(n)$ and $CSE(n)$, respectively, where

$$U(n) := \{U \in \mathcal{M}_{n,n}(\mathbb{C}) : UU^\dagger = I\},$$
$$COE(n) := \{U \in U(n) : U = U^t\},$$
$$CSE(n) := \{U \in U(2n) : U = U^D\}.$$

Remark 2.4. For convenience, let

$$\Omega_I^D(m,n) := \mathcal{M}_{m,n}(\mathbb{C}), \quad \Omega_{II}^D(n) := \mathcal{S}_n(\mathbb{C}), \quad \Omega_{III}^D(n) := \mathcal{A}_n(\mathbb{C}) \text{ or } \Omega_{III}^D(2n) := \mathcal{D}_n(\mathbb{C}).$$

3. Jack function of the matrix argument

In this section, we discuss the definition of the Jack function of the matrix argument and its propositions, to prepare for the following section.

Since power sum symmetric functions p_λ form a basis of symmetric polynomials, then for any Jack symmetric function $P_\lambda^{(\alpha)}$, there exist constants $a_{\lambda\mu}$ such that

$$P_\lambda^{(\alpha)} = \sum_\mu a_{\lambda\mu} p_\mu. \tag{3.1}$$

For any square matrix A and partition $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots)$, we set

$$\mathbf{p}_\lambda(A) := \prod_j (\mathrm{Tr}\, A^j)^{m_j}, \tag{3.2}$$

where $\mathrm{Tr}\, A$ stands for the trace of the matrix A. For $\alpha = 1, 2$, let

$$\mathbf{P}_\lambda^{(\alpha)}(A) := \sum_\mu a_{\lambda\mu} \mathbf{p}_\mu(A). \tag{3.3}$$

For $\alpha = \frac{1}{2}$, setting

$$\mathbf{P}_\lambda^{(\alpha)}(A) := \sum_\mu a_{\lambda\mu} \frac{1}{2^{|\mu|}} \mathbf{p}_\mu(A). \tag{3.4}$$

$\mathbf{P}_\lambda^{(\alpha)}(A)$ is called the Jack function of the matrix argument. By (2.38), we have

$$K_\lambda^{(\alpha)}(A, \overline{B}) = \frac{\alpha^{|\lambda|}}{d'_\lambda(\alpha)} \mathbf{P}_\lambda^{(\alpha)}(AB^\dagger) \tag{3.5}$$

or

$$K_\lambda^{(\alpha)}(A, B) = \frac{\alpha^{|\lambda|}}{d'_\lambda(\alpha)} \mathbf{P}_\lambda^{(\alpha)}(AB^t), \tag{3.6}$$

where $A, B \in \mathcal{M}_{m,n}$, or $A, B \in \mathcal{S}_n(\mathbb{C})$, or $A, B \in \mathcal{A}_n(\mathbb{C})$ and $\alpha = \frac{2}{a} = 1, 2, 4$, respectively.

Remark 3.1. For $\alpha = 1$, we have

$$\mathbf{P}_\lambda^{(\alpha)}(A) = S_\lambda(A). \tag{3.7}$$

Remark 3.2. Let $\{t_1, t_2, \ldots, t_n\}$ be eigenvalues of a square matrix A and λ be a partition of length $\leqslant n$, then

$$\mathbf{P}_\lambda^{(\alpha)}(A) := P_\lambda^{(\alpha)}(t_1, t_2, \ldots, t_n), \tag{3.8}$$

where $\alpha = 1, 2$.

Remark 3.3. Let A be a self-dual matrix of order $2n$, namely $A = A^D := Z_{2n}^{-1} X^t Z_{2n}$, where $Z_{2n} := I_n \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then eigenvalues of the matrix A can be written as $\{t_1, t_1, t_2, t_2, \ldots, t_n, t_n\}$, and we have

$$\mathbf{P}_\lambda^{(\alpha)}(A) = P_\lambda^{(\alpha)}(t_1, t_2, \ldots, t_n), \tag{3.9}$$

where $\alpha = \frac{1}{2}$.

By (2.1)–(2.3), we have Cauchy identity

$$\sum_\lambda \frac{h_\lambda(\alpha)}{d'_\lambda(\alpha)} \mathbf{P}_\lambda^{(\alpha)}(AB^\dagger)\mathbf{P}_\lambda^{(\alpha)}(CD^\dagger) = \begin{cases} \dfrac{1}{\det(I - AB^\dagger \otimes CD^\dagger)^{\frac{1}{\alpha}}}, & A, B, C, D \in \Omega_I \text{ or } \Omega_{II}, \alpha = 1, 2, \\ \dfrac{1}{\det(I - AB^\dagger \otimes CD^\dagger)^{\frac{1}{4\alpha}}}, & A, B, C, D \in \Omega_{III}, \alpha = \frac{1}{2}, \end{cases} \tag{3.10}$$

dual Cauchy identity

$$\sum_\lambda \mathbf{P}_\lambda^{(\alpha)}(AB^\dagger)\mathbf{P}_{\lambda'}^{(\frac{1}{\alpha})}(CD^\dagger) = \begin{cases} \det(I + AB^\dagger \otimes CD^\dagger), & \alpha = 1, \\ \sqrt{\det(I + AB^\dagger \otimes CD^\dagger)}, & A, B \in \Omega_{II} \text{ and } C, D \in \Omega_{III}, \alpha = 2, \end{cases} \tag{3.11}$$

and generalized binomial summation formula

$$\frac{1}{N(A, \overline{B})^s} = \sum_\lambda \frac{\alpha^{|\lambda|}[s]_\lambda^{(\alpha)}}{d'_\lambda(\alpha)} \mathbf{P}_\lambda^{(\alpha)}(AB^\dagger), \qquad A, B \in \Omega, \alpha = \frac{2}{a}. \tag{3.12}$$

Proposition 3.1. *Let Ω be Cartan domain, and $\mathcal{K}$ for stabilizer of the origin in the identity connected component of the group of biholomorphic self-maps of Ω. If ρ is a $\mathcal{K}$ invariant Borel probability measure on Ω^D, i.e. for any $k \in \mathcal{K}$,*

$$\mathrm{d}\rho(k \cdot Z) = \mathrm{d}\rho(Z).$$

Then we have

$$\int_{\Omega^D} K_\lambda^{(\alpha)}(Z_1, Z)\overline{K_\mu^{(\alpha)}(Z_2, Z)}\,\mathrm{d}\rho(Z) = \delta_{\lambda\mu}\frac{K_\lambda^{(\alpha)}(Z_1, \overline{Z_2})}{d_\lambda}\int_{\Omega^D} K_\lambda^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z) \tag{3.13}$$

and

$$\int_{\Omega^D} K_\lambda^{(\alpha)}(k \cdot Z, \overline{Z})\,\mathrm{d}\rho(Z) = \frac{\chi_\lambda(k)}{d_\lambda}\int_{\Omega^D} K_\lambda^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z), \tag{3.14}$$

where $d_\lambda := \dim \mathcal{P}_\lambda = \chi_\lambda(I)$, $\int_{\Omega^D} K_\lambda^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z) < +\infty$ and $\int_{\Omega^D} K_\mu^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z) < +\infty$.

Proof. By [6], we have that under the action $(\pi(k))f(Z) := f(k \cdot Z)$ of $\mathcal{K}$, the Hilbert space $L_a^2(\Omega^D, \rho)$ of square integrable analytic functions with respect to the measure ρ on Ω^D admits an irreducible decomposition

$$L_a^2(\Omega^D, \rho) = \bigoplus_{\int_{\Omega^D} K_\lambda^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z) < +\infty} \mathcal{P}_\lambda, \tag{3.15}$$

where $\bigoplus$ denotes the orthogonal direct sum.

For any partition λ of length $\leqslant r$, let $\{\varphi_i^\lambda, 1 \leqslant i \leqslant d_\lambda\}$ be a normal orthogonal basis of $\mathcal{P}_\lambda$ with respect to the Fock–Fischer inner product. We have

$$K_\lambda^{(\alpha)}(Z_1, \overline{Z_2}) = \sum_{i=1}^{d_\lambda} \varphi_i^\lambda(Z_1)\overline{\varphi_i^\lambda(Z_2)}. \tag{3.16}$$

By Schur's lemma, there exist positive constants c_λ that $\left\{c_\lambda\varphi_i^\lambda, 1 \leqslant i \leqslant d_\lambda\right\}$ is a normal orthogonal basis of $\mathcal{P}_\lambda$ with respect to the measure ρ when $\int_{\Omega^D} K_\lambda^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z) < +\infty$. Using

$$\int_{\Omega^D} c_\lambda\varphi_i^\lambda(Z)\overline{c_\mu\varphi_j^\mu(Z)}\,\mathrm{d}\rho(Z) = \delta_{\lambda\mu}\delta_{ij}, \tag{3.17}$$

and (3.16), we have

$$c_\lambda^2 \int_{\Omega^D} K_\lambda^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z) = d_\lambda. \tag{3.18}$$

Therefore

$$\begin{aligned}
\int_{\Omega^D} K_\lambda^{(\alpha)}(Z_1, Z)\overline{K_\mu^{(\alpha)}(Z_2, Z)}\,\mathrm{d}\rho(Z) &= \sum_{i=1}^{d_\lambda}\sum_{j=1}^{d_\mu} \varphi_i^\lambda(Z_1)\overline{\varphi_j^\mu(Z_2)} \int_{\Omega^D} \varphi_i^\lambda(Z)\overline{\varphi_j^\mu(Z)}\,\mathrm{d}\rho(Z) \\
&= \frac{1}{c_\lambda^2}\sum_{i=1}^{d_\lambda}\sum_{j=1}^{d_\mu} \varphi_i^\lambda(Z_1)\overline{\varphi_j^\mu(Z_2)}\delta_{\lambda\mu}\delta_{ij} \int_{\Omega^D} c_\lambda\varphi_i^\lambda(Z)\overline{c_\lambda\varphi_i^\lambda(Z)}\,\mathrm{d}\rho(Z) \\
&= \delta_{\lambda\mu}\frac{1}{c_\lambda^2}\sum_{i=1}^{d_\lambda} \varphi_i^\lambda(Z_1)\overline{\varphi_i^\lambda(Z_2)} = \delta_{\lambda\mu}\frac{1}{c_\lambda^2}K_\lambda^{(\alpha)}(Z_1, \overline{Z_2}) \\
&= \delta_{\lambda\mu}\frac{K_\lambda^{(\alpha)}(Z_1, \overline{Z_2})}{d_\lambda}\int_{\Omega^D} K_\lambda^{(\alpha)}(Z, \overline{Z})\,\mathrm{d}\rho(Z).
\end{aligned}$$

This proves (3.13).

Let

$$\left(\varphi_1^\lambda(k\cdot Z), \varphi_2^\lambda(k\cdot Z), \ldots, \varphi_{d_\lambda}^\lambda(k\cdot Z)\right) = \left(\varphi_1^\lambda(Z), \varphi_2^\lambda(Z), \ldots, \varphi_{d_\lambda}^\lambda(Z)\right)\pi_\lambda(k), \tag{3.19}$$

where $\pi_\lambda(k)$ are $d_\lambda \times d_\lambda$ matrices, and traces of $\pi_\lambda(k)$ equal to $\chi_\lambda(k)$. By (3.16) and (3.19), we have

$$K_\lambda^{(\alpha)}(k\cdot Z, \overline{Z}) = \mathrm{Tr}\left\{\pi_\lambda(k)\left(\varphi_1^\lambda(Z), \varphi_2^\lambda(Z), \ldots, \varphi_{d_\lambda}^\lambda(Z)\right)^\dagger\left(\varphi_1^\lambda(Z), \varphi_2^\lambda(Z), \ldots, \varphi_{d_\lambda}^\lambda(Z)\right)\right\}. \tag{3.20}$$

Using (3.17), we get

$$\int_{\Omega^D} \left(\varphi_1^\lambda(Z), \varphi_2^\lambda(Z), \ldots, \varphi_{d_\lambda}^\lambda(Z)\right)^\dagger\left(\varphi_1^\lambda(Z), \varphi_2^\lambda(Z), \ldots, \varphi_{d_\lambda}^\lambda(Z)\right)\mathrm{d}\rho(Z) = \frac{1}{c_\lambda^2}I_{d_\lambda}, \tag{3.21}$$

where I_{d_λ} are identity matrices. Finally, by (3.18), (3.20) and (3.21), we obtain (3.14). □

Corollary 3.1. *Let ρ be an invariant Borel probability measure on $\mathcal{M}_{m,n}(\mathbb{C})$, i.e. for any $U \in U(m)$, $V \in U(n)$*

$$\mathrm{d}\rho(UZV) = \mathrm{d}\rho(Z).$$

Then for any $Z_1, Z_2 \in \mathcal{M}_{m,n}(\mathbb{C})$, we have

$$\int_{\mathcal{M}_{m,n}(\mathbb{C})} S_\lambda(Z_1Z^t)\overline{S_\mu(Z_2Z^t)}\,\mathrm{d}\rho(Z) = \delta_{\lambda\mu}\frac{S_\lambda\left(Z_1Z_2^\dagger\right)}{d_\lambda}\int_{\mathcal{M}_{m,n}(\mathbb{C})} S_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \tag{3.22}$$

and for any $A \in \mathcal{M}_{m,m}(\mathbb{C})$, $B \in \mathcal{M}_{n,n}(\mathbb{C})$

$$\int_{\mathcal{M}_{m,n}(\mathbb{C})} S_\lambda(AZBZ^\dagger)\,\mathrm{d}\rho(Z) = \frac{S_\lambda(A)S_\lambda(B)}{d_\lambda}\int_{\mathcal{M}_{m,n}(\mathbb{C})} S_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \quad (3.23)$$

where $d_\lambda = S_\lambda(I_m)S_\lambda(I_n)$, $\int_{\mathcal{M}_{m,n}(\mathbb{C})} S_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$ *and* $\int_{\mathcal{M}_{m,n}(\mathbb{C})} S_\mu(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$.

Corollary 3.2. *Let* ρ *be an invariant Borel probability measure on* $\mathcal{S}_n(\mathbb{C})$*, i.e. for any* $U \in U(n)$,

$$\mathrm{d}\rho(UZU^t) = \mathrm{d}\rho(Z).$$

Then for any $Z_1, Z_2 \in \mathcal{S}_n(\mathbb{C})$*, we have*

$$\int_{\mathcal{S}_n(\mathbb{C})} \mathbf{P}^{(2)}_\lambda(Z_1 Z)\overline{\mathbf{P}^{(2)}_\mu(Z_2 Z)}\,\mathrm{d}\rho(Z) = \delta_{\lambda\mu}\frac{\mathbf{P}^{(2)}_\lambda\big(Z_1 Z_2^\dagger\big)}{d_\lambda}\int_{\mathcal{S}_n(\mathbb{C})} \mathbf{P}^{(2)}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \quad (3.24)$$

and for any $A \in \mathcal{M}_{n,n}(\mathbb{C})$,

$$\int_{\mathcal{S}_n(\mathbb{C})} \mathbf{P}^{(2)}_\lambda(AZA^tZ^\dagger)\,\mathrm{d}\rho(Z) = \frac{S_{2\lambda}(A)}{d_\lambda}\int_{\mathcal{S}_n(\mathbb{C})} \mathbf{P}^{(2)}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \quad (3.25)$$

where $d_\lambda = S_{2\lambda}(I_n)$, $\int_{\mathcal{S}_n(\mathbb{C})} \mathbf{P}^{(2)}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$ *and* $\int_{\mathcal{S}_n(\mathbb{C})} \mathbf{P}^{(2)}_\mu(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$.

Corollary 3.3. *Let* ρ *be an invariant Borel probability measure on* $\mathcal{A}_n(\mathbb{C})$*, i.e. for any* $U \in U(n)$,

$$\mathrm{d}\rho(UZU^t) = \mathrm{d}\rho(Z).$$

Then for any $Z_1, Z_2 \in \mathcal{A}_n(\mathbb{C})$*, we have*

$$\int_{\mathcal{A}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(Z_1 Z)\overline{\mathbf{P}^{(\frac{1}{2})}_\mu(Z_2 Z)}\,\mathrm{d}\rho(Z) = \delta_{\lambda\mu}\frac{\mathbf{P}^{(\frac{1}{2})}_\lambda\big(Z_1 Z_2^\dagger\big)}{d_\lambda}\int_{\mathcal{A}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \quad (3.26)$$

and for any $A \in \mathcal{M}_{n,n}(\mathbb{C})$,

$$\int_{\mathcal{A}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(AZA^tZ^\dagger)\,\mathrm{d}\rho(Z) = \frac{S_{\lambda\cup\lambda}(A)}{d_\lambda}\int_{\mathcal{A}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \quad (3.27)$$

where $d_\lambda = S_{\lambda\cup\lambda}(I_n)$, $\int_{\mathcal{A}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$ *and* $\int_{\mathcal{A}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\mu(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$.

Corollary 3.4. *Let* ρ *be an invariant Borel probability measure on space* $\mathcal{D}_n(\mathbb{C})$*, i.e. for any* $U \in U(2n)$,

$$\mathrm{d}\rho(UZU^D) = \mathrm{d}\rho(Z).$$

Then for any $Z_1, Z_2 \in \mathcal{D}_n(\mathbb{C})$*, we have*

$$\int_{\mathcal{D}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(Z_1 Z^t)\overline{\mathbf{P}^{(\frac{1}{2})}_\mu(Z_2 Z^t)}\,\mathrm{d}\rho(Z) = \delta_{\lambda\mu}\frac{\mathbf{P}^{(\frac{1}{2})}_\lambda\big(Z_1 Z_2^\dagger\big)}{d_\lambda}\int_{\mathcal{D}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \quad (3.28)$$

and for any $A \in \mathcal{M}_{2n,2n}(\mathbb{C})$,

$$\int_{\mathcal{D}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(AZA^DZ^\dagger)\,\mathrm{d}\rho(Z) = \frac{S_{\lambda\cup\lambda}(A)}{d_\lambda}\int_{\mathcal{D}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z), \quad (3.29)$$

where $d_\lambda = S_{\lambda\cup\lambda}(I_{2n})$, $\int_{\mathcal{D}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\lambda(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$ *and* $\int_{\mathcal{D}_n(\mathbb{C})} \mathbf{P}^{(\frac{1}{2})}_\mu(ZZ^\dagger)\,\mathrm{d}\rho(Z) < +\infty$.

Remark 3.4. Corollaries 3.1–3.4 will be useful for evaluating integrals over complex matrices ensembles.

Corollary 3.5. *Let Ω be classical domains of first, second and third types, and $\mathcal{K}$ the stabilizer of the origin in the identity connected component of the group of biholomorphic self-maps of Ω. Let $\mathrm{d}U$ be a unique $\mathcal{K}$ invariant Borel probability measure on the Shilov boundary $\partial_0\Omega$ of Ω, i.e. for any $k \in \mathcal{K}$,*

$$\mathrm{d}U(k\cdot Z) = \mathrm{d}U(Z).$$

Then we have

$$\int_{\partial_0\Omega} \mathbf{P}_\lambda^{(\alpha)}(Z_1U^t)\overline{\mathbf{P}_\mu^{(\alpha)}(Z_2U^t)}\,\mathrm{d}U = \delta_{\lambda\mu}\frac{\mathbf{P}_\lambda^{(\alpha)}\big(Z_1Z_2^\dagger\big)P_\lambda^{(\alpha)}(1_r)}{d_\lambda} \tag{3.30}$$

and

$$\int_{\partial_0\Omega} \mathbf{P}_\lambda^{(\alpha)}((k\cdot U)U^\dagger)\,\mathrm{d}U = \frac{\chi_\lambda(k)P_\lambda^{(\alpha)}(1_r)}{d_\lambda}, \tag{3.31}$$

where $d_\lambda = \dim \mathcal{P}_\lambda = \chi_\lambda(I)$, $\alpha = \frac{2}{a}$ and r is the rank of Ω.

Let

$$C\beta E(n) := \begin{cases} COE(n), & \beta = 1,\\ U(n), & \beta = 2,\\ CSE(n), & \beta = 4 \end{cases} \tag{3.32}$$

and

$$A^{\phi(\alpha)} := \begin{cases} A, & \alpha = 1,\\ A^t, & \alpha = 2,\\ A^D, & \alpha = \frac{1}{2}. \end{cases} \tag{3.33}$$

Using

$$\frac{e'_\lambda(\alpha,n)b_\lambda(\alpha,n)}{d'_\lambda(\alpha)h_\lambda(\alpha)} = \begin{cases} S_\lambda^2(I_n), & \alpha = 1,\\ S_{2\lambda}(I_n), & \alpha = 2,\\ S_{\lambda\cup\lambda}(I_{2n}), & \alpha = \frac{1}{2}. \end{cases} \tag{3.34}$$

We have the following.

Corollary 3.6. *Let $\mathrm{d}U$ be a unique invariant Borel probability measure on $C\beta E(n)$, $\alpha = \frac{2}{\beta}$, and λ, μ be partitions of length $\leqslant n$. We have*

$$\int_{C\beta E(n)} \mathbf{P}_\lambda^{(\alpha)}(AU)\overline{\mathbf{P}_\mu^{(\alpha)}(BU)}\,\mathrm{d}U = \delta_{\lambda\mu}\frac{d'_\lambda(\alpha)}{e'_\lambda(\alpha,n)}\mathbf{P}_\lambda^{(\alpha)}(AB^\dagger), \tag{3.35}$$

where A, B are square matrices such that $A = A^{\phi(\alpha)}$, $B = B^{\phi(\alpha)}$.

For any $n\times n$ (or $2n\times 2n$, when $\alpha = \frac{1}{2}$) matrix A,

$$\int_{C\beta E(n)} \mathbf{P}_\lambda^{(\alpha)}(AUA^{\phi(\alpha)}U^\dagger)\,\mathrm{d}U = \frac{d'_\lambda(\alpha)h_\lambda(\alpha)}{e'_\lambda(\alpha,n)b_\lambda(\alpha,n)}P_\lambda^{(\alpha)}(1_n)\times\begin{cases} S_\lambda^2(A), & \beta = 2,\\ S_{2\lambda}(A), & \beta = 1,\\ S_{\lambda\cup\lambda}(A), & \beta = 4. \end{cases} \tag{3.36}$$

Remark 3.5. Corollary 3.6 will be used for evaluating integrals over the circular ensembles below.

4. Integrals over the circular ensembles

Let Ω be a Cartan domain in $\Omega^D := \mathbb{C}^d$ in its Harish–Chandra realization. Let $t_1, t_2, \ldots, t_r$ be the singular values of $Z \in \Omega^D$, we define

$$\mathrm{Det}(Z, \overline{Z}) := \prod_{j=1}^{r} t_j^2. \tag{4.1}$$

Remark 4.1.

$$\mathrm{Det}(Z, \overline{Z}) = \begin{cases} \det(ZZ^\dagger), & Z \in \mathcal{M}_{m,n}(\mathbb{C}), \\ \det(ZZ^\dagger), & Z \in \mathcal{S}_n(\mathbb{C}), \\ \sqrt{\det(ZZ^\dagger)}, & Z \in \mathcal{A}_{2n}(\mathbb{C}) \text{ or } \mathcal{D}_{2n}(\mathbb{C}). \end{cases}$$

For real numbers s_1 and s_2, setting

$$\mathrm{d}\rho^D_{(s_1,s_2)}(Z) := \frac{1}{C} \frac{\mathrm{Det}(Z, \overline{Z})^{s_1}}{N_+(Z, \overline{Z})^{s_1+s_2+g}} \, \mathrm{d}m(Z), \tag{4.2}$$

$$\mathrm{d}\rho_{(s_1,s_2)}(Z) := \frac{1}{C} \mathrm{Det}(Z, \overline{Z})^{s_1} N(Z, \overline{Z})^{s_2} \, \mathrm{d}m(Z), \tag{4.3}$$

where g is the genus of Ω, $\rho^D_{(s_1,s_2)}$ is a probability measure on Ω^D, $\rho_{(s_1,s_2)}$ is a probability measure on Ω and the normalization C is given by

$$C = \int_{\Omega} \mathrm{Det}(Z, \overline{Z})^{s_1} N(Z, \overline{Z})^{s_2} \, \mathrm{d}m(Z) = \int_{\Omega^D} \frac{\mathrm{Det}(Z, \overline{Z})^{s_1}}{N_+(Z, \overline{Z})^{s_1+s_2+g}} \, \mathrm{d}m(Z). \tag{4.4}$$

In order to prove propositions 4.1 and 4.2, we need lemmas 4.1 and 4.2.

Lemma 4.1. *If $s_1 + b > -1$, $s_2 > -1$ and $\ell(\lambda) \leqslant r$, then*

$$\int_{\Omega^D} K^{(\alpha)}_\lambda(Z, \overline{Z}) \, \mathrm{d}\rho^D_{(s_1,s_2)}(Z) = \frac{\left[s_1 + \frac{d}{r}\right]^{(\alpha)}_\lambda}{(-1)^{|\lambda|}[-s_2]^{(\alpha)}_\lambda} \frac{d_\lambda}{\left[\frac{d}{r}\right]^{(\alpha)}_\lambda} = \frac{\alpha^{2|\lambda|}}{d'_{\lambda'}\left(\frac{1}{\alpha}\right) d'_\lambda(\alpha)} \frac{\left[s_1 + \frac{d}{r}\right]^{(\alpha)}_\lambda \left[\frac{r}{\alpha}\right]^{(\alpha)}_\lambda}{[\alpha s_2]^{(\frac{1}{\alpha})}_{\lambda'}} \tag{4.5}$$

and

$$\int_{\Omega} K^{(\alpha)}_\lambda(Z, \overline{Z}) \, \mathrm{d}\rho_{(s_1,s_2)}(Z) = \frac{\left[s_1 + \frac{d}{r}\right]^{(\alpha)}_\lambda}{[s_1 + s_2 + g]^{(\alpha)}_\lambda} \frac{d_\lambda}{\left[\frac{d}{r}\right]^{(\alpha)}_\lambda} = \frac{\alpha^{2|\lambda|}}{d'_\lambda(\alpha) h_\lambda(\alpha)} \frac{\left[s_1 + \frac{d}{r}\right]^{(\alpha)}_\lambda \left[\frac{r}{\alpha}\right]^{(\alpha)}_\lambda}{[s_1 + s_2 + g]^{(\alpha)}_\lambda}, \tag{4.6}$$

where $\alpha = \frac{2}{a}$ and for $a = 1, 2, 4$,

$$\begin{aligned} d_\lambda = K^{(\alpha)}_\lambda(e, \overline{e}) \left[\frac{d}{r}\right]^{(\alpha)}_\lambda &= \frac{\alpha^{2|\lambda|}}{d'_\lambda(\alpha) h_\lambda(\alpha)} \left[\frac{d}{r}\right]^{(\alpha)}_\lambda \left[\frac{r}{\alpha}\right]^{(\alpha)}_\lambda \\ &= \frac{\alpha^{2|\lambda|}}{d'_\lambda(\alpha) h_\lambda(\alpha)} \left[\frac{r-1}{\alpha} + b + 1\right]^{(\alpha)}_\lambda \left[\frac{r}{\alpha}\right]^{(\alpha)}_\lambda. \end{aligned} \tag{4.7}$$

Proof. (i) First prove (4.7). Using definition of $K^{(\alpha)}_\lambda(Z, \overline{Z})$, we have

$$d_\lambda := \dim \mathcal{P}_\lambda = \frac{\int_{\Omega^D} K^{(\alpha)}_\lambda(Z, \overline{Z}) \exp\{-|Z|^2\} \, \mathrm{d}m(Z)}{\int_{\Omega^D} \exp\{-|Z|^2\} \, \mathrm{d}m(Z)}. \tag{4.8}$$

We recall the formula for the integration on polar coordinates [4]:

$$\int_{\Omega^D} f(Z)\,\mathrm{d}m(Z) = c\int_{[0,+\infty)^r} 2^r \prod_{j=1}^r t_j^{2b+1} \prod_{1\leqslant j<k\leqslant r} |t_j^2 - t_k^2|^a \prod_{j=1}^r \mathrm{d}t_j \int_{\mathcal{K}} f\Big(k\cdot \sum_{j=1}^r t_j e_j\Big)\,\mathrm{d}k, \tag{4.9}$$

where c is a constant whose value can be found in [12]. From (4.9) and (2.35), we obtain

$$\int_{\Omega^D} K_\lambda^{(\alpha)}(Z,\overline{Z})\exp\{-|Z|^2\}\,\mathrm{d}m(Z)$$
$$= c\int_{[0,+\infty)^r} K_\lambda^{(\alpha)}\left(\sum_{j=1}^r t_j e_j, \overline{e}\right)\prod_{j=1}^r t_j^b \exp\{-t_j\}\prod_{1\leqslant j<k\leqslant r}|t_j - t_k|^a \prod_{j=1}^r \mathrm{d}t_j \tag{4.10}$$

and

$$\int_{\Omega^D} \exp\{-|Z|^2\}\,\mathrm{d}m(Z) = c\int_{[0,+\infty)^r} \prod_{j=1}^r t_j^b \exp\{-t_j\}\prod_{1\leqslant j<k\leqslant r}|t_j - t_k|^a \prod_{j=1}^r \mathrm{d}t_j. \tag{4.11}$$

By (2.25), (2.26) and (2.38), we have

$$\text{rhs of (4.10)} = K_\lambda^{(\alpha)}(e,\overline{e})\left[\frac{d}{r}\right]_\lambda^{(\alpha)}\cdot c\int_{[0,+\infty)^r}\prod_{j=1}^r t_j^b\exp\{-t_j\}\prod_{1\leqslant j<k\leqslant r}|t_j - t_k|^a\prod_{j=1}^r \mathrm{d}t_j. \tag{4.12}$$

Comparing (4.11) and (4.12), using (2.6) and (2.38), we obtain (4.7).

Now prove (4.5) and (4.6). Using (2.7), (2.26), (2.35) and (4.9), we obtain

$$\int_{\Omega^D} K_\lambda^{(\alpha)}(Z,\overline{Z})\frac{\mathrm{Det}(Z,\overline{Z})^{s_1}}{N_+(Z,\overline{Z})^{s_1+s_2+g}}\,\mathrm{d}m(Z)$$
$$= c\int_{[0,+\infty)^r} K_\lambda^{(\alpha)}\left(\sum_{j=1}^r t_j e_j,\overline{e}\right)\prod_{j=1}^r \frac{t_j^{s_1+b}}{(1+t_j)^{s_1+s_2+g}}\prod_{1\leqslant j<k\leqslant r}|t_j - t_k|^a\prod_{j=1}^r \mathrm{d}t_j$$
$$= K_\lambda^{(\alpha)}(e,\overline{e})\frac{\left[s_1+\frac{d}{r}\right]_\lambda^{(\alpha)}}{(-1)^{|\lambda|}[-s_2]_\lambda^{(\alpha)}}\cdot c\int_{[0,+\infty)^r}\prod_{j=1}^r\frac{t_j^{s_1+b}}{(1+t_j)^{s_1+s_2+g}}\prod_{1\leqslant j<k\leqslant r}|t_j - t_k|^a\prod_{j=1}^r \mathrm{d}t_j$$
$$= K_\lambda^{(\alpha)}(e,\overline{e})\frac{\left[s_1+\frac{d}{r}\right]_\lambda^{(\alpha)}}{(-1)^{|\lambda|}[-s_2]_\lambda^{(\alpha)}}\int_{\Omega^D}\frac{\mathrm{Det}(Z,\overline{Z})^{s_1}}{N_+(Z,\overline{Z})^{s_1+s_2+g}}\,\mathrm{d}m(Z). \tag{4.13}$$

Since

$$(-1)^{|\lambda|}[-s]_\lambda^{(\alpha)} = \frac{1}{\alpha^{|\lambda|}}[\alpha s]_{\lambda'}^{(\frac{1}{\alpha})} \tag{4.14}$$

and

$$d'_{\lambda'}\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^{|\lambda|}}h_\lambda(\alpha), \tag{4.15}$$

from (4.7) and (4.13), we get (4.5).

In a similar way, using (2.4), we have

$$\int_{\Omega} K_\lambda^{(\alpha)}(Z,\overline{Z})\mathrm{Det}(Z,\overline{Z})^{s_1}N(Z,\overline{Z})^{s_2}\,\mathrm{d}m(Z)$$
$$= c\int_{[0,1]^r} K_\lambda^{(\alpha)}\left(\sum_{j=1}^r t_j e_j,\overline{e}\right)\prod_{j=1}^r t_j^{s_1+b}(1-t_j)^{s_2}\prod_{1\leqslant j<k\leqslant r}|t_j - t_k|^a\prod_{j=1}^r \mathrm{d}t_j$$

$$= K_\lambda^{(\alpha)}(e,\overline{e})\frac{\left[s_1+\frac{d}{r}\right]_\lambda^{(\alpha)}}{[s_1+s_2+g]_\lambda^{(\alpha)}}\cdot c\int_{[0,1]^r}\prod_{j=1}^{r}t_j^{s_1+b}(1-t_j)^{s_2}\prod_{1\leqslant j<k\leqslant r}|t_j-t_k|^a\prod_{j=1}^{r}\mathrm{d}t_j$$

$$= K_\lambda^{(\alpha)}(e,\overline{e})\frac{\left[s_1+\frac{d}{r}\right]_\lambda^{(\alpha)}}{[s_1+s_2+g]_\lambda^{(\alpha)}}\int_\Omega \mathrm{Det}(Z,\overline{Z})^{s_1}N(Z,\overline{Z})^{s_2}\,\mathrm{d}m(Z),$$

by (4.7), we obtain (4.6).

□

Lemma 4.2. *Let Ω be a Cartan domain with dimension d. For $Z_1, Z_2 \in \Omega$ and real numbers s_1, s_2, we have*

$$\int_{\partial_0\Omega} N_+(Z_1,U)^{s_1}\overline{N_+(Z_2,U)^{s_2}}\,\mathrm{d}U = \sum_{\ell(\lambda)\leqslant r}\frac{[-s_1]_\lambda^{(\alpha)}[-s_2]_\lambda^{(\alpha)}}{\left[\frac{d}{r}\right]_\lambda^{(\alpha)}}K_\lambda^{(\alpha)}(Z_1,\overline{Z_2}) \tag{4.16}$$

and

$$\int_{\partial_0\Omega}\frac{1}{N(Z_1,U)^{s_1}\overline{N(Z_2,U)^{s_2}}}\,\mathrm{d}U = \sum_{\ell(\lambda)\leqslant r}\frac{[s_1]_\lambda^{(\alpha)}[s_2]_\lambda^{(\alpha)}}{\left[\frac{d}{r}\right]_\lambda^{(\alpha)}}K_\lambda^{(\alpha)}(Z_1,\overline{Z_2}), \tag{4.17}$$

where $\mathrm{d}U$ is a unique $\mathcal{K}$ invariant Borel probability measure on the Shilov boundary $\partial_0\Omega$ of Ω, $\alpha = \frac{2}{a}$.

Proof. By

$$\partial_0\Omega = \{k\cdot e : k \in \mathcal{K}\}, \tag{4.18}$$

it follows that

$$K_\lambda^{(\alpha)}(U,\overline{U}) = K_\lambda^{(\alpha)}(e,\overline{e}), \qquad U \in \partial_0\Omega. \tag{4.19}$$

This gives

$$\int_{\partial_0\Omega} K_\lambda^{(\alpha)}(U,\overline{U})\,\mathrm{d}U = K_\lambda^{(\alpha)}(e,\overline{e}). \tag{4.20}$$

Using (4.7), we get

$$\int_{\partial_0\Omega} K_\lambda^{(\alpha)}(U,\overline{U})\,\mathrm{d}U = \frac{d_\lambda}{\left[\frac{d}{r}\right]_\lambda^{(\alpha)}}. \tag{4.21}$$

By the Faraut–Koranyi formula (2.33), we have

$$N_+(Z,U)^{s_1} = \sum_{\ell(\lambda)\leqslant r}[-s]_\lambda^{(\alpha)}(-1)^{|\lambda|}K_\lambda^{(\alpha)}(Z,U), \qquad Z\in\Omega, \quad U\in\partial_0\Omega, \tag{4.22}$$

applying (3.13) and (4.21), we have

$$\int_{\partial_0\Omega} N_+(Z_1,U)^{s_1}\overline{N_+(Z_2,U)^{s_2}}\,\mathrm{d}U$$

$$= \sum_{\ell(\lambda)\leqslant r}\sum_{\ell(\mu)\leqslant r}[-s_1]_\lambda^{(\alpha)}[-s_2]_\mu^{(\alpha)}(-1)^{|\lambda|+|\mu|}\delta_{\lambda\mu}\frac{K_\lambda^{(\alpha)}(Z_1,\overline{Z_2})}{d\lambda}\int_{\partial_0\Omega}K_\lambda^{(\alpha)}(U,\overline{U})\,\mathrm{d}U$$

$$= \sum_{\ell(\lambda)\leqslant r}\frac{[-s_1]_\lambda^{(\alpha)}[-s_2]_\lambda^{(\alpha)}}{\left[\frac{d}{r}\right]_\lambda^{(\alpha)}}K_\lambda^{(\alpha)}(Z_1,\overline{Z_2}).$$

This proves (4.16).

The same argument as above, we obtain (4.17). □

As a consequence of lemmas 4.1 and 4.2, we have our main results in this section.

Proposition 4.1. *For $p \in \mathbb{N}, q \in \mathbb{R}$, let r_i, a_i, b_i, d_i, g_i and N_i be ranks, characteristic multiplicities, dimensions, genera and generic norms of Cartan domains Ω_i, respectively, $1 \leqslant i \leqslant 2$. And such that $a_1 \in \{1, 2, 4\}, a_2 = \frac{4}{a_1}, r_2 = p$ and $p < q + 1$.*

For $A, B \in \Omega_1$, we have

$$\int_{\partial_0\Omega_1} N_1(-A, U)^p \overline{N_1(-B, U)^q}\, \mathrm{d}U = \int_{\Omega_2^D} \mathrm{Det}(I + AB^\dagger \otimes ZZ^\dagger)\, \mathrm{d}\rho^D_{(s_1,s_2)}(Z), \tag{4.23}$$

where

$$\mathrm{Det}(I + AB^\dagger \otimes ZZ^\dagger) := \sum_\lambda \mathbf{P}^{(\alpha)}_\lambda(AB^\dagger)\mathbf{P}^{(\frac{1}{\alpha})}_{\lambda'}(ZZ^\dagger), \alpha = \frac{2}{a_1}, \tag{4.24}$$

$$\mathrm{d}\rho^D_{(s_1,s_2)}(Z) := \frac{1}{C}\frac{\mathrm{Det}(Z, \overline{Z})^{s_1}}{N_2(-Z, \overline{Z})^{s_1+s_2+g_2}}\, \mathrm{d}m(Z), \tag{4.25}$$

$$C = \int_{\Omega_2^D} \frac{\mathrm{Det}(Z, \overline{Z})^{s_1}}{N_2(-Z, \overline{Z})^{s_1+s_2+g_2}}\, \mathrm{d}m(Z) = \int_{\Omega_2} \mathrm{Det}(Z, \overline{Z})^{s_1} N_2(Z, \overline{Z})^{s_2}\, \mathrm{d}m(Z). \tag{4.26}$$

$s_1 = \frac{2}{a_1}(q - p + 1) - b_2 - 1$, *and* $s_2 = r_1 - 1 + \frac{2}{a_1}(b_1 + 1)$.

Proof. By (3.5) and (4.24), we have

$$\mathrm{Det}(I + AB^\dagger \otimes ZZ^\dagger) = \sum_\lambda d'_\lambda(\alpha) d'_{\lambda'}\left(\frac{1}{\alpha}\right) K^{(\alpha)}_\lambda(A, \overline{B}) K^{(\frac{1}{\alpha})}_{\lambda'}(Z, \overline{Z}). \tag{4.27}$$

Using (4.5), we get

$$\begin{aligned}\int_{\Omega^D} K^{(\frac{1}{\alpha})}_{\lambda'}(Z, \overline{Z})\, \mathrm{d}\rho^D_{(s_1,s_2)}(Z) &= \frac{\alpha^{-2|\lambda|}}{d'_{\lambda'}\left(\frac{1}{\alpha}\right) d'_\lambda(\alpha)} \frac{\left[s_1 + \frac{d_2}{r_2}\right]^{(\frac{1}{\alpha})}_{\lambda'} [\alpha r_2]^{(\frac{1}{\alpha})}_{\lambda'}}{\left[\frac{s_2}{\alpha}\right]^{(\alpha)}_\lambda} \\ &= \frac{\alpha^{-2|\lambda|}}{d'_{\lambda'}\left(\frac{1}{\alpha}\right) d'_\lambda(\alpha)} \frac{[\alpha p]^{(\frac{1}{\alpha})}_{\lambda'} [\alpha q]^{(\frac{1}{\alpha})}_{\lambda'}}{\left[\frac{d_1}{r_1}\right]^{(\alpha)}_\lambda}.\end{aligned} \tag{4.28}$$

Substituting (4.28) into (4.27), we obtain

$$\int_{\Omega_2^D} \mathrm{Det}(I + AB^\dagger \otimes ZZ^\dagger)\, \mathrm{d}\rho^D_{(s_1,s_2)}(Z) = \sum_\lambda \alpha^{-2|\lambda|} K^{(\alpha)}_\lambda(A, \overline{B}) \frac{[\alpha p]^{(\frac{1}{\alpha})}_{\lambda'} [\alpha q]^{(\frac{1}{\alpha})}_{\lambda'}}{\left[\frac{d_1}{r_1}\right]^{(\alpha)}_\lambda}. \tag{4.29}$$

On the other hand, by (4.14) and (4.16), we have

$$\int_{\partial_0\Omega_1} N_1(-A, U)^p \overline{N_1(-B, U)^q}\, \mathrm{d}U = \sum_\lambda \alpha^{-2|\lambda|} K^{(\alpha)}_\lambda(A, \overline{B}) \frac{[\alpha p]^{(\frac{1}{\alpha})}_{\lambda'} [\alpha q]^{(\frac{1}{\alpha})}_{\lambda'}}{\left[\frac{d_1}{r_1}\right]^{(\alpha)}_\lambda}. \tag{4.30}$$

Comparing (4.29) and (4.30), the result follows.

□

Proposition 4.2. *For $p, q \in \mathbb{R}$, let r_i, a_i, b_i, d_i, g_i and N_i be ranks, characteristic multiplicities, dimensions, genera and generic norms of Cartan domains Ω_i, respectively, $1 \leqslant i \leqslant 2$. And such that $a_1 \in \{1, 2, 4\}, a_1 = a_2, r_2 = \frac{2p}{a_1} \in \mathbb{N}, p < q + \frac{a_1}{2}$, and $p + q < \frac{r_1}{2}a_1 + b_1 + 1$.*

For $A, B \in \Omega_1$, we have

$$\int_{\partial_0\Omega_1} \frac{1}{N_1(A,U)^p\overline{N_1(B,U)^q}}\,\mathrm{d}U = \int_{\Omega_2} \frac{1}{\mathrm{Det}(I - AB^\dagger \otimes ZZ^\dagger)^{\frac{a_1}{2}}}\,\mathrm{d}\rho_{(s_1,s_2)}(Z), \tag{4.31}$$

where

$$\mathrm{Det}(I - AB^\dagger \otimes ZZ^\dagger)^{-\frac{a_1}{2}} := \sum_\lambda \frac{h_\lambda(\alpha)}{d'_\lambda(\alpha)}\mathbf{P}_\lambda^{(\alpha)}(AB^\dagger)\mathbf{P}_\lambda^{(\alpha)}(ZZ^\dagger), \qquad \alpha = \frac{2}{a_1}, \tag{4.32}$$

$$\mathrm{d}\rho_{(s_1,s_2)}(Z) := \frac{1}{C}\,\mathrm{Det}(Z,\overline{Z})^{s_1}N_2(Z,\overline{Z})^{s_2}\,\mathrm{d}m(Z), \tag{4.33}$$

$$C = \int_{\Omega_2} \mathrm{Det}(Z,\overline{Z})^{s_1}N_2(Z,\overline{Z})^{s_2}\,\mathrm{d}m(Z). \tag{4.34}$$

$s_1 = q - p + \frac{a_2}{2} - b_2 - 1$, *and* $s_2 = \frac{r_1}{2}a_1 + b_1 - (p+q)$.

Proof. By (4.32) and (3.5), we have

$$\mathrm{Det}(I - AB^\dagger \otimes ZZ^\dagger)^{-\frac{a_1}{2}} = \sum_\lambda \alpha^{-2|\lambda|}d'_\lambda(\alpha)h_\lambda(\alpha)K_\lambda^{(\alpha)}(A,\overline{B})K_\lambda^{(\alpha)}(Z,\overline{Z}). \tag{4.35}$$

Using

$$s_1 + \frac{d_2}{r_2} = q, \qquad \frac{r_2}{\alpha} = p, \qquad s_1 + s_2 + g_2 = \frac{d_1}{r_1},$$

and (4.6), we get

$$\int_{\Omega_2} K_\lambda^{(\alpha)}(Z,\overline{Z})\,\mathrm{d}\rho_{(s_1,s_2)}(Z) = \frac{\alpha^{2|\lambda|}}{d'_\lambda(\alpha)h_\lambda(\alpha)}\frac{\left[s_1 + \frac{d_2}{r_2}\right]_\lambda^{(\alpha)}\left[\frac{r_2}{\alpha}\right]_\lambda^{(\alpha)}}{[s_1+s_2+g_2]_\lambda^{(\alpha)}} = \frac{\alpha^{2|\lambda|}}{d'_\lambda(\alpha)h_\lambda(\alpha)}\frac{[q]_\lambda^{(\alpha)}[p]_\lambda^{(\alpha)}}{\left[\frac{d_1}{r_1}\right]_\lambda^{(\alpha)}}. \tag{4.36}$$

Substituting (4.36) into (4.35), we obtain

$$\int_{\Omega_2} \frac{1}{\mathrm{Det}(I - AB^\dagger \otimes ZZ^\dagger)^{\frac{a_1}{2}}}\,\mathrm{d}\rho_{(s_1,s_2)}(Z) = \sum_\lambda K_\lambda^{(\alpha)}(A,\overline{B})\frac{[p]_\lambda^{(\alpha)}[q]_\lambda^{(\alpha)}}{\left[\frac{d_1}{r_1}\right]_\lambda^{(\alpha)}}. \tag{4.37}$$

On the other hand, by (4.17), we have

$$\int_{\partial_0\Omega_1} \frac{1}{N(A,U)^p\overline{N(B,U)^q}}\,\mathrm{d}U = \sum_\lambda \frac{[p]_\lambda^{(\alpha)}[q]_\lambda^{(\alpha)}}{\left[\frac{d_1}{r_1}\right]_\lambda^{(\alpha)}}K_\lambda^{(\alpha)}(A,\overline{B}), \tag{4.38}$$

Comparing (4.37) and (4.38), the result follows. □

For the convenience of readers, by (3.10) and (3.11), we give the explicit expressions of (4.23) and (4.31).

Corollary 4.3. *Let m, n, p and q be positive integer numbers such that $m \leqslant n$ and $p \leqslant q$. For any $A, B \in \mathcal{M}_{m,n}(\mathbb{C})$, we have*

$$\int_{\partial_0\Omega_I(m,n)} \det(I + AU^t)^p\overline{\det(I + BU^t)^q}\,\mathrm{d}U$$
$$= \frac{1}{C_1}\int_{\mathcal{M}_{p,q}(\mathbb{C})} \det(I + AB^\dagger \otimes ZZ^\dagger)\frac{1}{\det(I + ZZ^\dagger)^{n+p+q}}\,\mathrm{d}m(Z) \tag{4.39}$$

$$= \frac{1}{C_2}\int_{\mathcal{M}_{p,p}(\mathbb{C})} \det(I + AB^\dagger \otimes ZZ^\dagger)\frac{\det(ZZ^\dagger)^{q-p}}{\det(I+ZZ^\dagger)^{n+p+q}}\,\mathrm{d}m(Z), \tag{4.40}$$

where C_1, C_2 are normalization constants.

Corollary 4.4. *Let m, n, p and q be positive integer numbers such that $m \leqslant n$, $p \leqslant q$ and $p + q \leqslant n$. For any $A, B \in \Omega_I(m, n)$, we have*

$$\int_{\partial_0\Omega_I(m,n)} \frac{1}{\det(I - AU^t)^p\overline{\det(I - BU^t)^q}}\,\mathrm{d}U$$

$$= \frac{1}{C_1}\int_{\Omega_I(p,q)} \frac{1}{\det(I - AB^\dagger \otimes ZZ^\dagger)}\det(I - ZZ^\dagger)^{n-p-q}\,\mathrm{d}m(Z) \tag{4.41}$$

$$= \frac{1}{C_2}\int_{\Omega_I(p,p)} \frac{1}{\det(I - AB^\dagger \otimes ZZ^\dagger)}\det(ZZ^\dagger)^{q-p}\det(I - ZZ^\dagger)^{n-p-q}\,\mathrm{d}m(Z), \tag{4.42}$$

where C_1, C_2 are normalization constants.

Corollary 4.5. *Let n and p be positive integer numbers, and q be a positive number such that $p < q + 1$. For any $A \in \mathcal{S}_n(\mathbb{C})$ and $B \in \Omega_{II}(n)$, we have*

$$\int_{COE(n)} \det(I + AU)^p\overline{\det(I + BU)^q}\,\mathrm{d}U$$

$$= \frac{1}{C_1}\int_{\mathcal{A}_{2p+1}(\mathbb{C})} \sqrt{\det(I + AB^\dagger \otimes ZZ^\dagger)}\frac{\det(ZZ^\dagger)^{q-p-\frac{1}{2}}}{\det(I + ZZ^\dagger)^{\frac{n}{2}+p+q}}\,\mathrm{d}m(Z) \tag{4.43}$$

$$= \frac{1}{C_2}\int_{\mathcal{A}_{2p}(\mathbb{C})} \sqrt{\det(I + AB^\dagger \otimes ZZ^\dagger)}\frac{\det(ZZ^\dagger)^{q-p+\frac{1}{2}}}{\det(I + ZZ^\dagger)^{\frac{n}{2}+p+q}}\,\mathrm{d}m(Z), \tag{4.44}$$

where C_1, C_2 are normalization constants.

Corollary 4.6. *Let n and p be positive integer numbers, and q be a positive number such that $p < q + \frac{1}{2}$ and $p + q < \frac{n}{2} + 1$. For any $A, B \in \Omega_{II}(n)$, we have*

$$\int_{COE(n)} \frac{1}{\det(I - AU)^p\overline{\det(I - BU)^q}}\,\mathrm{d}U$$

$$= \frac{1}{C}\int_{\Omega_{II}(2p)} \frac{1}{\sqrt{\det(I - AB^\dagger \otimes ZZ^\dagger)}}\det(ZZ^\dagger)^{q-p-\frac{1}{2}}\det(I - ZZ^\dagger)^{\frac{n}{2}-p-q}\,\mathrm{d}m(Z), \tag{4.45}$$

where C is a normalization constant.

Corollary 4.7. *Let n, q and p be positive integer numbers such that $p < q + 1$. For any $A, B \in \mathcal{D}_{2n}(\mathbb{C})$, we have*

$$\int_{CSE(n)} \det(I + AU^t)^{\frac{p}{2}}\overline{\det(I + BU^t)^{\frac{q}{2}}}\,\mathrm{d}U$$

$$= \frac{1}{C}\int_{\mathcal{S}_p(\mathbb{C})} \sqrt{\det(I + AB^\dagger \otimes ZZ^\dagger)}\frac{\det(ZZ^\dagger)^{\frac{1}{2}(q-p-1)}}{\det(I + ZZ^\dagger)^{\frac{1}{2}(p+q)+n}}\,\mathrm{d}m(Z), \tag{4.46}$$

where C is a normalization constant.

Corollary 4.8. *Let n and p be positive integer numbers, and q be a positive number such that $p < q + 2$ and $p + q < 2n + 1$. For any $A, B \in \Omega_{III}(2n)$, we have*

$$\int_{CSE(n)} \frac{1}{\det(I - AU^t)^{\frac{p}{2}} \overline{\det(I - BU^t)^{\frac{q}{2}}}} \mathrm{d}U$$
$$= \frac{1}{C} \int_{\Omega_{III}(2p)} \frac{1}{\sqrt{\det(I - AB^\dagger \otimes ZZ^\dagger)}} \det(ZZ^\dagger)^{\frac{1}{2}(q-p+1)} \det(I - ZZ^\dagger)^{n-\frac{1}{2}(p+q)} \mathrm{d}m(Z), \tag{4.47}$$

where C is a normalization constant.

5. Conclusions

In this paper, we investigate proof of the dual generalized Selberg formula, orthogonality of the Jack functions of matrices argument and relations of integrals over the circular ensembles to classical domains. In the unitary case ($\beta = 2$), these integrals have been the subject of earlier studies due to their relevance to random matrix theory and its applications [14, 17, 18]. And, by corollary 3.6 and Littlewood identities [3], we arrive at integrals

$$\int_{C\beta E(n)} \exp\{\delta(\beta) \operatorname{Tr}(AUA^{\phi(\alpha)}U^\dagger)\} \mathrm{d}U = \sum_\lambda \frac{1}{\left[1 + \frac{\beta}{2}(n-1)\right]_\lambda^{(\frac{2}{\beta})}} \times \begin{cases} S_\lambda^2(A), & \beta = 2, \\ S_{2\lambda}(A), & \beta = 1, \\ S_{\lambda\cup\lambda}(A), & \beta = 4, \end{cases} \tag{5.1}$$

$$\int_{C\beta E(n)} \exp\{\delta(\beta) \operatorname{Tr}(AU + BU^\dagger)\} \mathrm{d}U = {}_0F_1^{(\alpha)}\left(1 + \frac{\beta}{2}(n-1); AB\right), \tag{5.2}$$

and

$$\int_{C\beta E(n)} \frac{1}{\operatorname{Det}(I - AUA^{\phi(\alpha)}U^\dagger)^{1+\frac{\beta}{2}(n-1)}} \mathrm{d}U = \begin{cases} \frac{1}{\det(I - A\vee A)}, & \beta = 1, \\ \frac{1}{\det(I - A\otimes A)}, & \beta = 2, \\ \frac{1}{\det(I - A\wedge A)}, & \beta = 4, \end{cases} \tag{5.3}$$

where

$$\delta(\beta) := \begin{cases} 1, & \beta = 1, \\ 1, & \beta = 2, \\ \frac{1}{2}, & \beta = 4 \end{cases} \tag{5.4}$$

and when $\|A\| < 1$,

$$\operatorname{Det}(I - A) := \begin{cases} \det(I - A), & \beta = 1, \\ \det(I - A), & \beta = 2, \\ \sqrt{\det(I - A)}, & \beta = 4. \end{cases}$$

Generalized hypergeometric function of matrix argument ${}_0F_1^{(\alpha)}(s; A)$ defined by

$${}_0F_1^{(\alpha)}(s; A) := \sum_\lambda \frac{\alpha^{|\lambda|}}{d'_\lambda(\alpha)[s]_\lambda^{(\alpha)}} \mathbf{P}_\lambda^{(\alpha)}(A),$$

and symbols $A \vee B$, $A \otimes B$ and $A \wedge B$ stand for symmetric tensor product, tensor product and anti-symmetric tensor product for matrices A and B, respectively.

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